ON DECOMPOSABLE OPERATORS : Decomposability of multipliers on Banach algebras

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ON DECOMPOSABLE OPERATORS

(Decomposability of multipliers on Banach algebras)

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1. Introduction. Every multiplier on a semisimple commutative Banach algebra has the single-valued extension property but not necessarily decomposable. In fact J. Eshmeier [4] shows that there exists a multiplier on the group algebra of a non-discrete locally compact abelian group which is not decomposable. This is a negative solution of Colojoara-Foias' question [3].

The purpose of this talk is to investigate the decomposability of multipliers on a semisimple commutative Banach algebra. We show that every multiplier whose Gelfand transform is continuous in the hull-kernel topology is (strongly) decomposable. Applying this we can show the equivalence of the regularity of the algebra and the decomposability of the multiplication operators, which asserts the converse of Colojoara-Foias' result [3] holds. We further give many measures on a locally compact abelian group such that the corresponding multipliers on the group algebra are decomposable. The class of these measures contains the class of measures whose continuous parts are absolutely continuous and so our result contains Eschmeier's one given in [4]. The proof makes use of the natural embedding of measure algebras considered by J. Inoue [6].

2. Results. Let A be a semisimple commutative Banach algebra with carrier space Φ_A and $M(A)$ the multiplier algebra of A with carrier space $\Phi_{M(A)}$. We denote by $T^{\vee}$ the Gelfand representation of $T \in M(A)$ and denote by $T^{\wedge}$ the restriction of $T^{\vee}$ to Φ_A. Then our main result is the following

Theorem 1. Let $T \in M(A)$ be such that $T^{\vee}$ is continuous on $\Phi_{M(A)}$ in the hull-kernel topology and J be a T-invariant closed ideal of A. Then $T|J$ and T/J are decomposable and their spectral capacities are respectively given by $\ker((T|J)^{\wedge -1}(C\setminus F))$ and $\ker((T/J)^{\wedge -1}(C\setminus F))$ for each closed set F in the complex plane **C**.

The following result is rapidly obtained from the above theorem and [1, Theorem 1.7].

Corollary 2. Let T be as in the above theorem. Then T is strongly decomposable.

The following lemma gives a sufficient condition for continuity of $T^{\vee}$ in the hull-kernel topology.

Lemma 3. Let $T \in M(A)$. If $T^{\vee}$ is constant on $\Phi_{M(A)} \setminus \Phi_A$ and if $T^{\wedge}$ is continuous on Φ_A in the hull-kernel topology, then $T^{\vee}$ is continuous on $\Phi_{M(A)}$ in the hull-kernel topology.

The above results imply immediately the following result obtained by I. Colojoara and C. Foias [3]:

Corollary 4. If A is regular, then every multiplication operator on A is strongly decomposable.

A bounded linear operator T on a Banach space X is said to have the weak 2-spectral decomposition property (abbreviated SDP) if for any open covering $\sigma(T) \subset G_1 \cup G_2$ there are T-invariant subspaces X_1 and X_2 of X such that $X = (X_1 + X_2)^-$ and $\sigma(T | X_i) \subset G_i$ $(i = 1, 2)$.

Theorem 5. If $T \in M(A)$ has the weak 2-SDP, then $T^{\wedge}$ is continuous on Φ_A in the hull-kernel topology.

By combining Theorems 1, 5 and Lemma 3, we obtain the following characterization of the decomposability of multiplication operator:

Corollary 6. Let T be an arbitrary multiplication operator on A. Then the following four conditions are equivalent:

(i) T is strongly decomposable.

(ii) T is decomposable.

(iii) T has the weak 2-SDP.

(iv) $T^{\wedge}$ is continuous on Φ_A in the hull-kernel topology.

Therefore the above corollary implies immediately the following converse of Corollary 4:

Corollary 7. If every multiplication operator on A has the weak 2-SDP (in particular is decomposable), then A is a regular algebra.

In particular since the measure algebra $M(G)$ of a non-discrete locally compact abelian group G is not regular, we have the following result obtained by J. Eschmeier [4] : there exists a measure $\mu \in M(G)$ such that the convolution operator : $\nu \to \mu * \nu$ ($\nu \in M(G)$) does not posseses the weak 2-SDP.

Now let G be a locally compact abelian group and denote by top(G) the class of all locally compact group topologies on G which are equal or stronger than the original topology on G. For each $\tau \in$ top(G), let $\widetilde{L^1}(G, \tau)$ be the kernel of the hull of $L^1(G, \tau)$ in the measure algebra $M(G, \tau)$. Then $\widetilde{L^1}(G, \tau)$ can be regarded as a closed subalgebra of the measure algebra $M(G)$ of G (see [6]). Let $\widetilde{L^{\cdot}}(G)$ be the closed subalgebra of $M(G)$ generated by $\{\widetilde{L^1}(G, \tau) : \tau \in \text{top}(G)\}$.

The following result is essentially pointed out by J. Inoue.

Lemma 8. Under the above notation, every Gelfand transform of measure in $\widetilde{L^{*}}(G)$ is continuous on $\Phi_{M(G)}$ in the hull-kernel topology.

He actually proves it from the following general viewpoint:

Lemma 9. Let X be a commutative Banach algebra with identity and B a Banach subalgebra of X. If B is regular, then for any $b \in B$ its Gelfand transform $b^{\vee}$ (considered as $b \in X$) is continuous on Φ_X in the hull-kernel topology.

Therefore by combining Corollary 2 and Lemma 8 we have the following

Theorem 10. The multiplier : $f \to \mu * f$ on $L^1(G)$ is strongly decomposable for every measure μ in $\widetilde{L^{*}}(G)$.

This contains the following result obtained by J. Eschmeier [4]:

Corollary 12 (Eschmeier). The multiplier: $f \to \mu * f$ on $L^1(G)$ is strongly decomposable for every measure μ whose continuous part is absolutely continuous.

Remark. We can from [8, Theorem 2.10] observe that there are many locally compact abelian groups G such that the cardinality of top(G) $\geqq 3$. Then for such groups G, the class of measures on G whose continuous part are absolutly continuous is strictly contained $\widetilde{L^{*}}(G)$ from [6, Corollary 2.7].

3. Proofs. We have only to prove Theorems 1 and 5 and Lemmas 3, 8 and 9.

(1) Proof of Theorem 1. We shall prove the restricted case. The quotient case can be proved by the same method too. We first show that

$$J_T(F) = \ker((T|J)^{\wedge -1}(\mathbf{C} \setminus F))$$

for each closed set F in $\mathbf{C}$. Here $J_T(F)$ denotes the set of all $x \in J$ with $\sigma_{T|J}(x) \subset F$. To do this let F be any closed set in $\mathbf{C}$. If $x \in$

$J_T(F)$, then for each $\lambda \in \mathbf{C}\setminus F$, there is $x_\lambda \in J$ with $(\lambda - T)x_\lambda = x$, so that $x^\wedge(\varphi) = 0$ for all $\varphi \in (T|J)^{\wedge-1}(\mathbf{C}\setminus F)$. In other words, $J_T(F) \subset \ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$. It remains to prove the converse inclusion, but it is sufficient to prove

$$\mathbf{C}\setminus F \subset \rho(T|\ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F)).$$

To do this let $\lambda \in \mathbf{C}\setminus F$ be fixed. Since $\lambda - T^\wedge(\varphi) \neq 0$ for each $\varphi \in (T|J)^{\wedge-1}(F)$ and J is semisimple, it follows that $(\lambda - T)\,|\,\ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$ is injective. Thus it remains only to prove that $(\lambda - T)\,|\,\ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$ is surjective. To do this let $x \in \ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$ be fixed. Set

$$\delta = \inf\{\,|\lambda - \lambda'| : \lambda' \in F\}.$$

Then $\delta > 0$ and $|\lambda - T^\wedge(\varphi)| \geqq \delta$ for all $\varphi \in (T|J)^{\wedge-1}(F)$. Now put

$$H = \{\varphi \in \Phi_{M(A)} : |\lambda - T^\vee(\varphi)| \geqq \delta\}.$$

By the hull-kernel continuity of $T^\vee$, H is a hull in $\Phi_{M(A)}$. Then by [8, Theorem 3.6.15], there exists $S \in M(A)$ such that $((\lambda - T)S)^\vee(\varphi) = 1$ for all $\varphi \in H$. Set $y = Sx$. Then $y \in \ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$. Also observe that $((\lambda - T)y)^\wedge|\Phi_J = x^\wedge|\Phi_J$ and hence $(\lambda - T)y = x$ by the semisimplicity of J. We thus obtain that $(\lambda - T)\,|\,\ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F))$ is surjective.

We next show that $T|J$ is decomposable. To do this let G and H be a pair of open discs with $G^- \subset H$. Choose an open disc U with $G^- \subset U \subset U^- \subset H$. Then $T^{\vee-1}(G^-) \cap T^{\vee-1}(\mathbf{C}\setminus U) = \phi$. Also $T^{\vee-1}(G^-)$ and $T^{\vee-1}(\mathbf{C}\setminus U)$ are hulls in $\Phi_{M(A)}$ and $\Phi_{M(A)}$ is compact. Therefore by [8, Corollary 3.6.10], there exists $S \in M(A)$ such that $S^\vee\,|\,T^{\vee-1}(G^-) = 1$ and $S^\vee\,|\,T^{\vee-1}(\mathbf{C}\setminus U) = 0$. Set

$$F_1 = (T^\vee(T^{\vee-1}(\mathbf{C}\setminus G^-)))^-$$

and

$$F_2 = (T^\vee(\{\varphi \in \Phi_{M(A)} : S^\vee(\varphi) \neq 0\}))^-.$$

Then it is easily observed that $F_1 \subset \mathbf{C}\setminus G$ and $F_2 \subset H$. Moreover set

$$J_i = \ker((T|J)^{\wedge-1}(\mathbf{C}\setminus F_i)) \quad (i = 1, 2).$$

By the first argument, $J_i = J_T(F_i)$ $(i = 1, 2)$ and hence J_1 and J_2 are spectral maximal spaces of $T|J$ such that $\sigma(T|J_i) \subset F_i$ $(i = 1, 2)$ by [3, Proposition 1.3.8]. In this case, $J = J_1 + J_2$. In fact let $x \in J$ be fixed and set $x_1 = x - Sx$, $x_2 = Sx$, hence $x = x_1 + x_2$. By the definition of F_1, we have $(T|J)^{\vee -1}(\mathbf{C}\setminus F_1) \subset T^{\vee -1}(G^-)$, so that $x_1\hat{}(\varphi) = x\hat{}(\varphi) - S\hat{}(\varphi)x\hat{}(\varphi) = 0$ for all $\varphi \in (T|J)^{\hat{}-1}(\mathbf{C}\setminus F_1)$. Then $x_1 \in J_1$. By the definition of F_2, we have $S^\vee \mid T^{\vee -1}(\mathbf{C}\setminus F_2) = 0$, so that $x_2\hat{}(\varphi) = S\hat{}(\varphi)x\hat{}(\varphi) = 0$ for all $\varphi \in (T|J)^{\hat{}-1}(\mathbf{C}\setminus F_2)$. Then $x_2 \in J_2$. Therefore the desired conclusion follows from [7, Theorem 2.3]. Also since $J_T(F) = \ker((T|J)^{\hat{}-1}(\mathbf{C}\setminus F))$ for F closed in $\mathbf{C}$, it follows from [5] or [2] that the spectral capacity for $T|J$ is given by $\ker((T|J)^{\hat{}-1}(\mathbf{C}\setminus F))$ for each closed set F in $\mathbf{C}$. Q.E.D.

(2) Proof of Lemma 3. Without loss of generality we can assume that $T^\vee|\Phi_{M(A)}\setminus\Phi_A = 0$. Let G be an open set in $\mathbf{C}$ but fixed. If G does not contain 0, then $T^{\vee -1}(G) = T^{\hat{}-1}(G)$, so $T^{\vee -1}(G)$ is open in the hull-kernel topology from the hull-kernel continuity of $T\hat{}$ and the hull-kernel openness of Φ_A in $\Phi_{M(A)}$. Then we may assume that G contains 0. Choose $\varepsilon > 0$ such that $|\lambda| \geq \varepsilon$ for all $\lambda \in \mathbf{C}\setminus G$. Since $T^\vee|\Phi_{M(A)}\setminus\Phi_A = 0$, it follows that $T^{\vee -1}(\mathbf{C}\setminus G) \subset \{\varphi \in \Phi_A : |T\hat{}(\varphi)| \geq \varepsilon\}$. Then $T^{\vee -1}(\mathbf{C}\setminus G) = T^{\hat{}-1}(\mathbf{C}\setminus G)$ is a hull in Φ_A. Note also that $\{\varphi \in \Phi_A : |T\hat{}(\varphi)| \geq \varepsilon\}$ is compact, and hence so is $T^{\vee -1}(\mathbf{C}\setminus G)$. Then by [8, Theorem 3.6.7], there exists $e \in A$ such that $e\hat{}(\varphi) = 1$ for all $\varphi \in T^{\vee -1}(\mathbf{C}\setminus G)$. For each $a \in A$, let $\mu_a(x) = ax$, $x \in A$. Set $S = I - \mu_e$. Then $S \in M(A)$ with $S^\vee|T^{\vee -1}(\mathbf{C}\setminus G) = 0$. Let φ be any element of the hull-kernel closure of $T^{\vee -1}(\mathbf{C}\setminus G)$ in $\Phi_{M(A)}$. Since S belongs to the kernel of $T^{\vee -1}(\mathbf{C}\setminus G)$ in $M(A)$ it follows that $S^\vee(\varphi) = 0$ and hence $\mu_e{}^\vee(\varphi) = 1$, so $\varphi \in \Phi_A$. In this case $\varphi \in T^{\vee -1}(\mathbf{C}\setminus G)$. Otherwise, there is $x \in A$ such that $x\hat{}(\varphi) \neq 0$ and $x\hat{}|T^{\vee -1}(\mathbf{C}\setminus G) = 0$ since $T^{\vee -1}(\mathbf{C}\setminus G)$ is a hull in Φ_A. Then μ_x belongs to the kernel of $T^{\vee -1}(\mathbf{C}\setminus G)$ in $M(A)$, so that $\mu_x\hat{}(\varphi) = 0$, hence

$x^{\wedge}(\varphi) = 0$, a contradiction. Consequently $T^{\vee -1}(\mathbf{C}\setminus G)$ is a hull in $\Phi_{M(A)}$. In other words $T^{\vee -1}(G)$ is open in the hull-kernel topology.

Q.E.D.

(3) Proof of Theorem 5. Suppose that $T \in M(A)$ has the weak SDP. Let F be an arbitrary closed set in $\mathbf{C}$ and $E = T^{\wedge -1}(F)$. Suppose that φ_0 is in $\Phi_A\setminus E$ such that $\varphi_0|\ker(E) = 0$. Choose $a_0 \in A$ with $\varphi_0(a_0) = 1$ and set $\lambda_0 = T^{\wedge}(\varphi_0)$, so $\lambda_0 \in \mathbf{C}\setminus F$. Since $\{\mathbf{C}\setminus F, \mathbf{C}\setminus\{\lambda_0\}\}$ is an open covering of $\sigma(T)$ and T has the weak 2-SDP, it follows that there exist T-invariant subspaces A_1 and A_2 of A such that $A = (A_1 + A_2)^{-}$, $\sigma(T \mid A_1) \subset \mathbf{C}\setminus F$ and $\sigma(T \mid A_2) \subset \mathbf{C}\setminus\{\lambda_0\}$. Then there exist $a_1 \in A_1$ and $a_2 \in A_2$ such that $\|a_1 + a_2 - a_0\| < 1/2$. In this case $a_i^{\wedge}(\varphi_0) = 0$ $(i = 1, 2)$. Indeed let $\varphi \in E$. Then $T^{\wedge}(\varphi) \in F \subset \rho(T|A_1)$, and hence there is a bounded linear operator S_1 on A_1 such that $(T^{\wedge}(\varphi) - T|A_1)S_1 = id|A_1$. Therefore we have

$$\begin{aligned} a_1^{\wedge}(\varphi) &= ((T^{\wedge}(\varphi) - T)S_1(a_1))^{\wedge}(\varphi) \\ &= T^{\wedge}(\varphi)(S_1(a_1))^{\wedge}(\varphi) - (TS_1(a_1))^{\wedge}(\varphi) \\ &= 0, \end{aligned}$$

whence $a_1 \in \ker(E)$. Since $\varphi_0|\ker(E) = 0$, it follows that $a_1^{\wedge}(\varphi_0) = 0$. Also since $\lambda_0 \in \rho(T|A_2)$, there is a bounded linear operator S_2 on A_2 such that $(\lambda_0 - T|A_2)S_2 = id|A_2$. We then obtain $a_2^{\wedge}(\varphi_0) = 0$ by the same computation. We therefore have

$$1 = |a_1^{\wedge}(\varphi_0) + a_2^{\wedge}(\varphi_0) - a_0^{\wedge}(\varphi_0)| \leqq \|a_1 + a_2 - a_0\| < 1/2,$$

a contradiction. Consequently E must be closed in the hull-kernel topology.

Q.E.D.

(4) Proof of Lemma 9. We can assume without loss of generality that B contains the identity of X. Then it is sufficient to show that the restriction map $\pi : \Phi_X \to \Phi_B$; $\varphi \to \varphi|B$ is continuous in the hull-kernel topology. To do this let F be a closed subset of Φ_B in the hull-kernel topology. Then $\{\varphi \in \Phi_X : \varphi|\ker F = 0\} \subset \pi^{-1}(F)$. Also since $\ker F \subset \ker\pi^{-1}(F)$, it follows that $\mathrm{hull}(\ker\pi^{-1}(F)) \subset \{\varphi \in \Phi_X$

: $\varphi | \ker F = 0\}$. Therefore $\pi^{-1}(F)$ is closed in the hull-kernel topology. In other words, π is continuous in this topology.

(5) Proof of Lemma 8. Since $L^1(G, \tau)$ is regular, it follows from [10, Lemma 4.3 (ii)] that $\widetilde{L^1}(G, \tau)$ is also regular. Then the desired result follows from the preceding lemma.

4. Problems. It will be natural to propose the following two problems :

(1) Is a decomposable multiplier on A continuous on $\Phi_{M(A)}$ in the hull-kernel topology ?

(2) If every multiplier on A is decomposable, is the multiplier algebra M(A) of A is regular ?

We note that these questions are, by Corollaries 6 and 7, correct whenever A has an identity. If also (1) is true, then so is (2).

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